SINGLE-VALUEDNESS OF A GENERAL SOLUTION OF THE PROBLEM OF MOTION OF A HEAVY RIGID BODY IN A NEWTONIAN FORCE FIELD IN ONE PARTICULAR CASE

(OB ODNOZNACHNOSTI OBSHCHEGO RESHENIIA ZADACHI O DVIZHENII TIAZHELOGO TELA V N'IUTONOVSKOM POLE SIL V ODNOM SLUCHAE).

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It is well known that the equations of motion of a heavy rigid body about a fixed point in a Newtonian force field which are

$$A\frac{dp}{dt}+(C-B)\,qr=-Mg\,(y_0\gamma''-z_0\gamma')+\frac{3g}{R}(C-B)\,\gamma'\gamma'',\qquad \frac{d\gamma}{dt}=r\gamma'-q\gamma''$$
$$(ABC,\ \ pqr,\ \ \gamma\gamma'\gamma'',\ \ x_0y_0z_0)\qquad (0.1)$$

can be reduced to quadratures in the real time only in two cases [1–3]

$$x_0=0,\quad y_0=0,\quad z_0=0;\qquad A=B,\quad x_0=0,\quad y_0=0\qquad (0.2)$$

for which the fourth algebraic integral [4,5] exists.

We have shown in [6] that all the possible cases of single-valuedness of the general solution of (0.1), are equivalent to the two cases shown in (0.2). From this it follows that the proof of the single-valuedness of the general proof for the above two cases (0.2), will clarify the question of the possibility of existence of the following general theorem [7]:

The fourth algebraic integral of the system (0.1) exists if, and only if general solutions for p, q, r, γ, γ', γ'' exist, which are single-valued over the whole of the t-plane.

Proof of the single-valuedness of the general solution of (0.1) and of the remaining six direction cosines in the second of the (0.2) cases, follows.

1. The case under consideration analogous to the Lagrange case in the classical problem of the motion of a heavy, rigid body about a fixed point, has the following equations of motion:

$$\begin{gathered} dp\,/\,dt-mqr=-(\alpha+au)\gamma',\quad u=\gamma'' \\ dq\,/\,dt+mpr=(\alpha+au)\gamma,\qquad dr\,/\,dt=0 \\ d\gamma\,/\,dt=r\gamma'-qu\qquad (pqr,\ \gamma\gamma'\ u) \end{gathered}\qquad (1.1)$$

four independent first integrals of which are

$$p^2 + q^2 - 2\alpha u - au^2 = h, \quad p\gamma + q\gamma' - bu = k, \quad r = r_0, \quad \gamma^2 + \gamma'^2 + u^2 = 1$$
$$\alpha = -Mgz_0 / A, \quad m = (A - C) / A, \quad a = 3gm / R, \quad b = (m - 1) r_0 \qquad (1.2)$$

Introducing polar coordinates ρ and σ

$$p = \rho \cos \sigma, \quad q = \rho \sin \sigma, \quad \rho^2 = h + 2\alpha u + au^2 \qquad (1.3)$$

we obtain

$$(du / dt)^2 = (1 - u^2) \rho^2 - (k + bu)^2 \equiv \Phi(u)$$
$$d\sigma / dt = -r_0 + (L_1 u + L_2) / \rho^2, \; L_1 = ak - \alpha b, \quad L_2 = \alpha k - bh \qquad (1.4)$$

from which u and σ [3] can be determined.

From the first equation of (1.4) it follows that u can be found by transforming an elliptic integral and that it can, together with its derivative, be expressed in terms of single-valued functions of time.

Taking into account (1.3) let us, instead of p and q, consider the function $W = \rho e^{i\sigma}$.

In order to show that W is a single-valued function of time (from which the single-valuedness of p and q also follows), we shall now prove that the function $W_1 = e^{ir_0 t} W$ is a single-valued function of u and of du/dt, and we shall do it by investigating its behavior near its singular points. We shall represent W_1 by (1.4) and

$$W_1 = \rho e^{i\sigma_1}; \qquad \sigma_1 = \int F_1(u)\, du, \qquad F_1(u) = \frac{L_1 u + L_2}{\rho^2 \sqrt{\Phi(u)}} \qquad (1.5)$$

factorising the polynomials ρ^2 and $\Phi(u)$ thus

$$\rho^2 = a(u - \lambda_1)(u - \lambda_2), \qquad \Phi(u) = -a(u - \mu_1)(u - \mu_2)(u - \mu_3)(u - \mu_4) \; (1.6)$$

Since the single-valuedness of the solution when all the roots of $\Phi(u)$ are distinct implies the single-valuedness of the general solution of (1.1), we shall continue to assume that μ_l $(l = 1, 2, 3, 4)$ are all distinct.

2. The relationships between the roots of the polynomials λ_j, μ_l $(j = 1, 2; l = 1, 2, 3, 4)$ the values $u = \lambda_j$, $u = \mu_l$ of which are singular points of W_1, admit of the five following cases:

1°. Let $\lambda_1 \neq \lambda_2$, $\lambda_j \neq \mu_l$. In this case (1.4) yields

$$(L_1 u + L_2)\rho^{-2} = \sum_{j=1}^{j=2} {}^1/_2 (\kappa + b\lambda_j)(u - \lambda_j)^{-1}, \quad \Phi^{-1/2}(\lambda_j) = i\varepsilon (k + b\lambda_j)^{-1} \qquad (2.1)$$
$$k + b\lambda_j \neq 0, \qquad \operatorname{res}\,[F_1(u)]_{u=\lambda_j} = {}^1/_2\, i\varepsilon \qquad (\varepsilon = \pm 1, \; j = 1, 2)$$

From (1.5), (1.6) and (2.1) it follows that the points $u = \lambda_j$ are removable singular points, hence in their vicinity, W_1 is a single-valued function of u. Near each of the points μ_l $(l = 1, 2, 3, 4)$ the first equation of (1.4) will give us

$$F_1(u) = \xi_l^{-1/2} f_1(\xi_l); \quad \sigma_1 = \xi_l^{1/2} f_2(\xi_l) = \Phi^{1/2}(u) f_3(\xi_l) = f_3(\xi_l)(du / dt) \; (\xi_l = u - \mu_l) \qquad (2.2)$$

where $f_s(\eta)$ $(s = 1, 2, \ldots)$ are holomorphic functions of η.

From (1.5) and (2.2) it follows that in vicinity of the points $u = \mu_l$, W_1 is a single-valued function of u and du/dt. Since in the remaining cases the proof is almost identical with the one just given, we shall give it only when it differs from the one already given.

2°. Let $\lambda_1 = \lambda_2 = \lambda$, $\lambda \neq \mu_l$. At $u = \lambda$, we have

$$(L_1 u + L_2)\,\rho^{-2} = (k + \lambda b)\,(u - \lambda)^{-1}, \quad \Phi^{-1/2}(\lambda) = i\varepsilon\,(k + \lambda b)^{-1} \tag{2.3}$$

$$k + \lambda b \neq 0, \quad \rho^2 = a\,(u - \lambda)^2, \quad \mathrm{res}\,[F_1(u)]_{u=\lambda} = i\varepsilon, \quad \varepsilon = \pm 1 \tag{2.4}$$

3°. Let $\lambda_1 \neq \lambda_2$, $\lambda_1 = \mu_1$, $\lambda_2 \neq \mu_s$ $(s = 2, 3, 4)$. Here, by the first equation of (1.4), we have $b^2 + k^2 \neq 0$.

Near the point $u = \mu_1$, we shall have

$$F_1(u) = \xi_1^{-3/2} f_4(\xi_1); \quad \sigma_1 = \xi_1^{-1/2} f_5(\xi_1) = \Phi^{-1/2}(u)\, f_6(\xi_1) = f_6(\xi_1)\,(du/dt)^{-1} \tag{2.5}$$

and $$\rho = \xi_1^{1/2} f_7(\xi_1) = \Phi^{1/2}(u)\, f_8(\xi_1) = (du/dt)\, f_8(\xi_1) \qquad (\xi_1 = u - \mu_1) \tag{2.6}$$

4°. Let $\lambda_1 \neq \lambda_2$, $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$. In this case $b = k = 0$, and from the first equation of (1.4) together with (1.5) it follows that $\sigma_1 =$ const, while ρ near the points $u = \lambda_j$ $(j = 1, 2)$ will assume the form (2.6).

5°. Let $\lambda_1 = \lambda_2 = \lambda = \mu_1$. By (2.3) the expansion near the point $u = \lambda$, will be of the form (2.5).

Hence, p, q, r, $\gamma'' = u$ are single-valued functions of time, and by (1.1), γ and γ' have the same property.

3. We shall now show that the remaining six direction cosines are also single-valued functions of time. Using Eulerian angles θ, φ, ψ in the usual manner, it will be sufficient to show [8] that

$$\cos\varphi\cos\psi, \quad \cos\varphi\sin\psi, \sin\varphi\sin\psi, \sin\varphi\cos\psi, \sin\psi\sin\theta, \; \cos\psi\sin\theta \tag{3.1}$$

are single-valued. From γ, γ', γ''

$$\gamma = \sin\theta\sin\varphi, \qquad \gamma' = \sin\theta\cos\varphi, \qquad \gamma'' = \cos\theta$$

which were shown above to be single-valued functions of time,

$$\sin\varphi = V_1 \sin\theta, \qquad \cos\varphi = V_2 \sin\theta \tag{3.2}$$

follows, where V_s $(s = 1, 2, \ldots)$ are single-valued functions of time.

Instead of $\sin\theta\sin\psi$ and $\sin\theta\cos\psi$, let us consider

$$W_2 = \sqrt{1 - u^2}\, e^{i\psi} \tag{3.3}$$

where ψ by (1.2), can be determined from

$$d\psi / dt = (k + bu) / (1 - u^2) \tag{3.4}$$

We shall show that W_2 is a single-valued function of time. To do this, it will be sufficient to show that W_2 is a single-valued function of u and du/dt, where ψ, by the first equation of (1.4) and (3.4) is given by

$$\psi = \int F_2(u)\,du, \qquad F_2(u) = \frac{k + bu}{(1 - u^2)\sqrt{\Phi(u)}} \tag{3.5}$$

The proof differs from that given for W_1, in the following details:- in the formulas of the 1-st, 3-rd and 4-th case, λ_1 and λ_2 should be replaced by 1 and -1 respectively, σ_1 replaced by ψ, ρ by $\rho_1 = (1 - u^2)^{1/2}$, $F_1(u)$ by $F_2(u)$, L_1 by b and L_2 by k.

Hence W_2 is a single-valued function of time, and

$$\sin\psi = V_3 \sin\theta, \qquad \cos\psi = V_4 \sin\theta \tag{3.6}$$

From (3.2) and (3.6) it follows that (3.1) and hence the remaining six direction cosines, are single-valued functions of time.

It follows that in order to complete the existence proof of the theorem given above, the single-valuedness of the general solution in the first case of (0.2) with the condition $A \neq B$, remains to be investigated (for the case $A = B$, single-valuedness of the general solution follows from the results obtained above).

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